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$\mathcal{A}$	an algorithm . . . . .	56, 83
$A \sqsubseteq B$	$A$ is simpler than $B$ . . . . .	97
$\mathcal{C}$	a cell in an arrangement of hyperplanes . . . . .	57, 96
$\Delta$	the maximum outdegree of a partition graph . . . . .	81
$\varepsilon$	a generic infinitesimal . . . . .	49
	a sufficiently small positive real number . . . . .	25, 72
	an arbitrarily positive constant (in an asymptotic time bound) . . . . .	62
$f^*$	the dual flat of $f$ . . . . .	94
$I(P, H)$	the number of incidences between points $P$ and hyperplanes $H$ . . . . .	69
$I_d(n, m)$	maximum number of incidences between $n$ points and $m$ hyperplanes in $\mathbb{R}^d$ . . . . .	69
$i \perp j$	$i$ and $j$ are relatively prime . . . . .	72
$I(P, H)$	the number of incidences between points $P$ and hyperplanes $H$ . . . . .	100
$K(\varepsilon)$	an extension of the ordered field $K$ . . . . .	49
$\tilde{K}$	the real closure of the ordered field $K$ . . . . .	49
$\log^* n$	iterated logarithm of $n$ . . . . .	11
$M(P, H)$	the relative orientation matrix of points $P$ and hyperplanes $H$ . . . . .	68
$\mu(P, H)$	minimum monochromatic cover size of points $P$ and hyperplanes $H$ . . . . .	69
$\mu_d(n, m)$	worst case monochromatic cover size for $n$ points and $m$ hyperplanes in $\mathbb{R}^d$ . . . . .	69
$\mu_d^\circ(n, m)$	worst case monochromatic cover size for $n$ points and $m$ hyperplanes in $\mathbb{R}^d$ with no incidences . . . . .	69
$[n]$	the integers $\{1, 2, \dots, n\}$ . . . . .	72
$n^{\underline{a}}$	falling factorial power: $n!/(n-a)!$ . . . . .	58
$\Omega(\cdot)$	asymptotic lower bound . . . . .	2

$\omega_d(t)$	the d-dimensional weird moment curve: $(t, t^2, \dots, t^{d-1}, t^{d+1})$ . . . . .	20
$\phi$	a fixed linear expression in $r$ variables . . . . .	50
$\varphi(n)$	the Euler totient function . . . . .	72
$\Phi \leq \Pi$	$\Phi$ is a face of $\Pi$ . . . . .	94
$\pi_r(P, H)$	minimum $r$ -polyhedral cover size of points $P$ and hyperplanes $H$ . . . . .	100
$\hat{\pi}_{d,r}(n, m)$	worst case $r$ -polyhedral cover size for monochromatic configurations of $n$ points and $m$ hyperplanes in $\mathbb{R}^d$ . . . . .	101
$\pi_{d,r}^\circ(n, m)$	worst case $r$ -polyhedral cover size of $n$ points and $m$ hyperplanes in $\mathbb{R}\mathbb{P}^d$ with no incidences . . . . .	100
$\Pi^*$	the dual of the projective polyhedron $\Pi$ . . . . .	95
$\text{proj}_f(X)$	the projection of a set $X$ by a flat $f$ . . . . .	95
$\mathcal{Q}_A$	the set of query polynomials used by algorithm $\mathcal{A}$ . . . . .	56
$\mathbb{R}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$	a tower of field extensions of the reals . . . . .	49
$\mathbb{R}\mathbb{P}^d$	$d$ -dimensional real projective space . . . . .	94
$\mathcal{R}_v$	the set of query regions associated with a node $v$ in a partition graph . . . . .	81
$\sigma$	the inner product doubling map from $\mathbb{R}^3$ to $\mathbb{R}^6$ . . . . .	80
$\sigma_d$	the inner product doubling map from $\mathbb{R}^{d+1}$ to $\mathbb{R}^{\binom{d+2}{2}}$ . . . . .	101
$\text{span}(X)$	the projective span of $X$ . . . . .	94
$\text{susp}_f(X)$	the suspension of a set $X$ by a flat $f$ . . . . .	95
$T_A(P, H)$	the running time of algorithm $\mathcal{A}$ given $P$ and $H$ as input . . . . .	83
$\mathcal{U}$	an open set in $(\mathbb{R}\mathbb{P}^d)^n$ . . . . .	98
$\#W$	number of connected components of a set $W$ . . . . .	5
$\zeta(P, H)$	minimum zero cover size of points $P$ and hyperplanes $H$ . . . . .	69
$\zeta_d(n, m)$	worst case zero cover size for $n$ points and $m$ hyperplanes in $\mathbb{R}^d$ . . . . .	69

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