Kink-free deformations of polygons

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Abstract

We consider a discrete version of the Whitney-Graustein theorem concerning regular equivalence of closed curves. Two regular polygons P and P', i.e. polygons without overlapping adjacent edges, are called regularly equivalent if there is a continuous one-parameter family $P_s, 0 \leq s \leq 1$, of regular polygons with $P_0 = P$ and $P_1 = P'$. Geometrically the one-parameter family is a kink-free deformation transforming P into P'. The winding number of a polygon is a complete invariant of its regular equivalence class. We develop a linear algorithm that determines a linear number of elementary steps to deform a regular polygon into any other regular polygon with the same winding number.

1 Introduction

The Whitney-Graustein theorem states that up to regular ('kink-free') deformation a regular closed curve in the plane is completely determined by its winding number. In this paper we consider the discrete version of this theorem in which closed curves are replaced with polygons. A setting for this problem is given in Section 2. In Section 3 we show how to reduce a polygon to isothetic form in which every edge is parallel to one of the coordinate axes. Section 4 contains the algorithm that reduces an isothetic polygon to isothetic normal form that is uniquely determined by its winding number. In Section 5 we indicate possible extensions of this work to the classification of polygonal curves on piecewise flat surfaces. This paper was inspired by similar work of Mehlhorn and Yap, cf. [MY]. They derive an algorithm that computes a quadratic number of elementary steps to transform between regularly equivalent polygons. The sequence of steps can be determined in linear time. Their normal form as well as their method is however completely different from ours.

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2 Equivalence of Regular Polygons

A closed curve C in the plane is called *regular* if it has a smooth parametrization $f : [0,1] \to \mathbb{R}^2$ such that $f(0) = f(1), f'(0) = f'(1)$ and $f'(t) \neq 0$ for $0 \leq t \leq 1$. The *winding number* $w(C)$ of C is the degree of the map $f_* : S^1 \to S^1$ defined by $f_*(t) = f'(t)/|f'(t)|$. The regularity of f allows us to consider t as a parameter on the unit-circle S^1. Geometrically speaking the winding number of C is the number of full turns — counted with sign — of the tangent of C as we go around C. The figure-of-eight has winding number 0, a regularly embedded circle has winding number ± 1, depending on its orientation, see Figure 1.

Two regular closed curves C and C' are called regularly equivalent if there is a continuous family $C_s, 0 \leq s \leq 1$, of regular closed curves with $C_0 = C$ and $C_1 = C'$. Since the winding number of C_s varies continuously with s and has integer value a *necessary* condition for regular equivalence of C and C' is $w(C) = w(C')$. In [W] it is proved that this condition is also *sufficient*. This result is usually referred to as the Whitney-Graustein theorem. As illustrated in [MY] any deformation of the figure-of-eight into the unit circle has to go via a curve containing a *kink*, i.e.

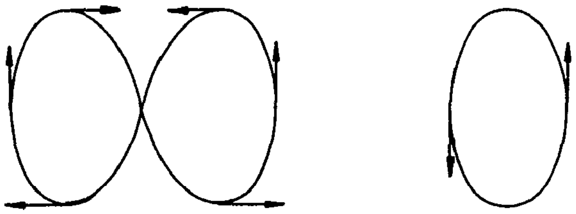

Figure 1

a point without a well-defined tangent.

A—not necessarily simple—polygon P is represented by its sequence of vertices $p_0, \ldots, p_{n-1}$. Two sequences of vertices are called equivalent if one can be obtained from the other by a cyclic shift. The polygon P corresponding to an equivalence class is denoted by $P = (p_0, \ldots, p_{n-1})$. P is called *regular* if for any triple p_{i-1}, p_i, p_{i+1} of consecutive vertices p_{i-1} and p_{i+1} are not on the same half-line with origin p_i. Here and in the sequel addition is modulo the number of vertices of the polygon. The (exterior) ***angle*** φ_i of P at vertex p_i is the ***signed*** angle between the directed lines $l(p_{i-1}, p_i)$ and $l(p_i, p_{i+1})$, see Figure 2. The *winding number* $w(P)$ of P is $\left(\sum_{0 \leq i < n} \varphi_i\right) / (2\pi)$. It is easy to check that $w(P)$ is an integer and that $w(P) = \pm 1$ for a ***simple*** polygon.

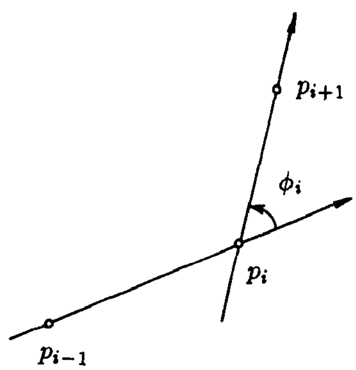

Figure 2

Two polygons with equal winding number may have a different number of vertices. Therefore the class of admissible deformations must include insertion and deletion of vertices. The deformations are of three types, cf. [MY].

(T0) *Insertion.* The regular polygon $P = (p_0, \ldots, p_{n-1})$ may be transformed into $P' = (p_0, \ldots, p_i, q, p_{i+1}, \ldots, p_{n-1}), 0 \leq i < n$, where q is a point lying strictly inbetween p_i and p_{i+1}, i.e. $q = (1-t)p_i + tp_{i+1}$ for some t with $0 < t < 1$.

(T1) *Deletion.* The regular polygon $P = (p_0, \ldots, p_{n-1})$ may be transformed into $P' = (p_0, \ldots, p_{i-1}, p_{i+1}, \ldots, p_{n-1}), 0 \leq i < n$, provided p_{i-1}, p_i and p_{i+1} are collinear.

(T2) *Translation.* The regular polygon $P = (p_0, \ldots, p_{n-1})$ may be transformed into $P' = (p_0, \ldots p_{i-1}, p'_i, p_{i+1}, \ldots, p_{n-1}), 0 \leq i < n$, provided the polygon $P_t = (p_0, \ldots, p_{i-1}, (1-t)p_i + tp'_i, p_{i+1}, \ldots, p_{n-1})$ is regular for $0 \leq t \leq 1$.

Our definition of $T2$ is different from the one in [MY], but is equivalent to it. It is easy to see that a transformation of type $T0$, $T1$ or $T2$ does not affect the winding number.

Definition 2.1 *Two regular polygons P and P' are called equivalent if P can be transformed into P' by a finite sequence of transformations of type $T0, T1$ and $T2$.*

This relation obviously is a true equivalence relation. Our main result is:

Theorem 2.2 *Let P and P' be regular polygons with n and n' vertices, respectively. Then P and P' are equivalent if and only if $w(P) = w(P')$. Moreover if $w(P) = w(P')$ there is a sequence of transformations of type $T0, T1$ and $T2$ of length $O(n+n')$ establishing the equivalence between P and P'. This sequence can be determined in $O(n+n')$ time.*

Mehlhorn and Yap prove a similar result although their sequence of transformations has *quadratic* length. The algorithm reduces a polygon to a normal form that is uniquely determined by the winding number. Since the class of elementary steps is closed under inversion we can deform the normal form into any other polygon with the same winding number. In [MY] this normal form is a ***star-polygon.*** In this paper we introduce the ***isothetic normal form*** which leads to a completely different—and in our point of view simpler—method of reduction.

3 Transformation to an Isothetic Polygon

We consider the plane to be endowed with a fixed system of orthogonal x, y-coordinates. The unit vectors along the axes are denoted by e_0 and e_1, respectively. For vectors v and w the inner product is denoted by $< v, w >$. We consider a vector to be determined up to translations, i.e. we don't fix its 'begin-point'. A polygon P is called *isothetic* if any edge is parallel to the x- or y-axis, and *reduced isothetic* if successive edges are perpendicular.

Lemma 3.1 *Let P be a regular polygon with n vertices. There is a sequence of transformations of type $T0, T1$ and $T2$ of length $O(n)$ transforming P into a reduced isothetic polygon with at most $3n$ vertices. The sequence can be determined in $O(n)$ time.*

Before giving the proof we introduce some terminology. A *chain-sequence* is a finite sequence of vectors in the plane. A chain-sequence is called *isothetic* if all its elements are parallel to one of the coordinate axes, and *reduced isothetic* if any pair of successive elements is moreover perpendicular. Two chain-sequences are called equivalent if one sequence can be obtained from the other by a cyclic shift. We abbreviate reduced isothetic polygon (chain-sequence) by RIP (RIC). Given a point p in the plane there is a one-one correspondence between the set of isothetic polygonal curves starting at p and the set of chain-sequences: if $(v_0, \dots, v_{n-1})$ is a chain-sequence the polygonal curve $p_0, \dots, p_n$ is defined by $p_0 = p$, $p_{i+1} = p_i + v_i$ for $0 \le i < n$. Similarly any RIP is determined up to a translation by an equivalence class of RIC's whose elements add up to the zero vector.

Consider a vector v whose projections v_0 and v_1 onto the x- and y-axis are non-zero. The isothetic chain-sequences $L(v), R(v), LR(v)$ and $RL(v)$ are defined as (see Figure 3):

$(v_1, v_0), (v_0, v_1), (\frac{1}{2}v_1, v_0, \frac{1}{2}v_1), (\frac{1}{2}v_0, v_1, \frac{1}{2}v_0)$, resp.
if $< v_0, e_0 >$ and $< v_1, e_1 >$ have the same sign;
$(v_0, v_1), (v_1, v_0), (\frac{1}{2}v_0, v_1, \frac{1}{2}v_0), (\frac{1}{2}v_1, v_0, \frac{1}{2}v_1)$, resp.
if $< v_0, e_0 >$ and $< v_1, e_1 >$ have different sign.

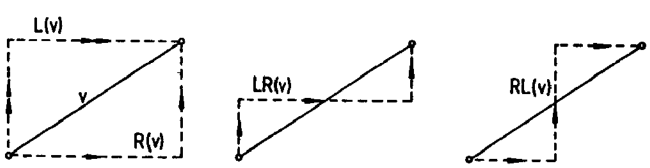

Figure 3

Proof of Lemma 3.1: Let $(v_0, \dots, v_{n-1})$ be a chain-sequence representing P and let $\overline{v}_j$ be the (open) edge corresponding to v_j. Assuming the chain-sequence of P is isothetic up to the tail $(v_i, \dots, v_{n-1})$, for some i with $0 \le i < n$, we indicate how to transform P into P' such that the chain-sequence of P' is isothetic up to the tail $(v_{i+1}, \dots, v_{n-1})$. If v_i is parallel to the x- or y-axis we take $P' = P$. Otherwise let T_l be the closed triangular region bounded by the polygonal curves with chain-sequences (v_i) and $L(v_i)$ whose begin-point coincides with that of $\overline{v}_i$. The chain-sequence $c(v_i)$ is defined as

$$\begin{array}{rll} L(v_i) & ,\text{if} & \overline{v}_{i-1} \cap T_l = \emptyset \text{ and } \overline{v}_{i+1} \cap T_l = \emptyset; \\ R(v_i) & ,\text{if} & \overline{v}_{i-1} \cap T_l \neq \emptyset \text{ and } \overline{v}_{i+1} \cap T_l \neq \emptyset; \\ LR(v_i) & ,\text{if} & \overline{v}_{i-1} \cap T_l = \emptyset \text{ and } \overline{v}_{i+1} \cap T_l \neq \emptyset; \\ RL(v_i) & ,\text{if} & \overline{v}_{i-1} \cap T_l \neq \emptyset \text{ and } \overline{v}_{i+1} \cap T_l = \emptyset. \end{array}$$

Let P' be the polygon obtained by replacing v_i in the chain-sequence of P with $c(v_i)$. It is not hard to check that P' is obtained from P by at most two transformations of type $T0$ followed by the same number of transformations of type $T2$. We refer to Figure 4 for an illustration of the case $\overline{v}_{i+1} \cap T_l \neq \emptyset$.

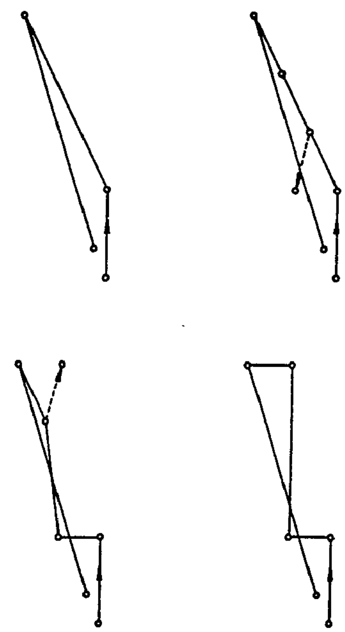

Figure 4

Since we introduce at most two vertices per edge of P the isothetic polygon into which P is transformed has at most $3n$ vertices. Deleting in linear time all vertices shared by collinear adjacent edges we obtain a RIP equivalent to P. □

4 Reduction to Normal Form

4.1 Transforming RIC's

We shall reduce a RIP to normal form via a finite sequence of transformations of its chain-sequence.

Since the class of RIP's is not invariant under transformations of type $T0$, $T1$ and $T2$, we replace them with transformations leaving this class invariant.

For a vector v parallel to $e_i, i = 0$ or $i = 1$, we define $\text{sign}(v) = \text{signum}< v, e_i >$. We introduce three types of transformations on RIC's. Again the modifications are essentially local.

(IT0) An equivalence class C of RIC's with representative $(v_0, \ldots, v_{n-1})$ may be transformed into C' with representative $(v_0, \ldots, v'_{i-1}, v_i, v'_{i+1}, \ldots, v_{n-1})$ provided

- v_{i-1} and v'_{i-1} as well as v_{i+1} and v'_{i+1} are parallel;
- $v'_{i-1} + v'_{i+1} = v_{i-1} + v_{i+1}$
- $\text{sign}(v'_{i-1}) = \text{sign}(v_{i-1})$ and $\text{sign}(v'_{i+1}) = \text{sign}(v_{i+1})$

(IT1) An equivalence class C of RIC's with representative $(v_0, \ldots, v_{n-1})$ may be transformed into C' with representative $(v_0, \ldots, v_{i-1}, \lambda_0 v_i, \lambda_1 v_{i+1}, (1-\lambda_0)v_i, (1-\lambda_1)v_{i+1}, v_{i+2}, \ldots, v_{n-1})$ provided $(\lambda_0, \lambda_1) \in \Lambda_0 \cup \Lambda_1 \cup \Lambda_2$, where

$$\begin{aligned}
\Lambda_0 &= \{(\lambda_0, \lambda_1) | \quad 0 < \lambda_0 < 1, \quad 0 < \lambda_1 < 1\}, \\
\Lambda_1 &= \{(\lambda_0, \lambda_1) | \quad 0 < \lambda_0 < 1, \quad \lambda_1 < 0\}, \\
\Lambda_2 &= \{(\lambda_0, \lambda_1) | \quad \lambda_0 > 1, \quad 0 < \lambda_1 < 1\}.
\end{aligned}$$

The numbers λ_0, λ_1 will be called the *parameters* of the transformation. See Figure 5.

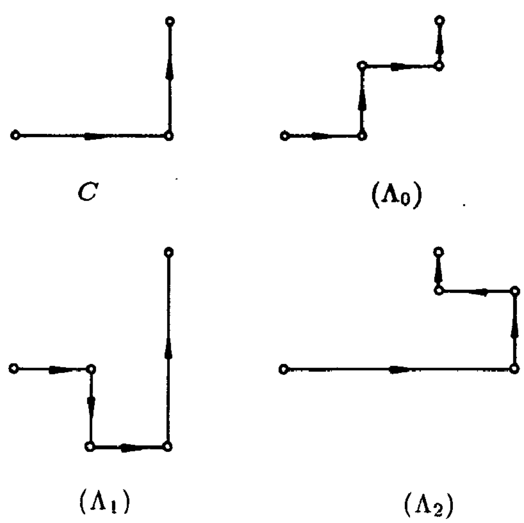

Figure 5

(IT2) An equivalence class C of *RIC's* may be transformed into C' if there is a transformation of type $IT1$ taking C' into C.

Lemma 4.1 *Let P be a RIP with chain-sequence C. Let C' be a RIC obtained by applying a transformation of type $IT0$, $IT1$ or $IT2$ to C. There is a sequence of at most four transformations of type $T0$, $T1$ and $T2$ taking P into a RIP with chain-sequence C'.*

Proof: Let $C = (v_0, \ldots, v_{n-1})$ and let $p_0, \ldots, p_{n-1}$ be the vertices of P such that $p_i = p_{i-1} + v_{i-1}$. We merely consider a transformation of type $IT1$ with parameters $(\lambda_0, \lambda_1) \in \Lambda_1$. So suppose $C' = (v_0, \ldots, v_{i-1}, \lambda_0 v_i, \lambda_1 v_{i+1}, (1-\lambda_0)v_i, (1-\lambda_1)v_{i+1}, v_{i+2}, \ldots, v_{n-1})$. Let $p' = p_i + \lambda_0 v_i, p'' = p_i + \frac{1}{2}(1+\lambda_0)v_i, q' = p' + \lambda_1 v_{i+1}, q'' = q_1 + (1-\lambda_0)v_i$. Note that $q'' + (1-\lambda_1)v_{i+1} = p_{i+2}$. The polygon $P' = (p_0, \ldots, p_i, p', q', q'', p_{i+2}, \ldots, p_{n-1})$ has chain-sequence C'. It is obtained from P via two transformations of type $T0$—insertion of p' and p''—followed by two transformations of type $T2$, see Figure 6. □

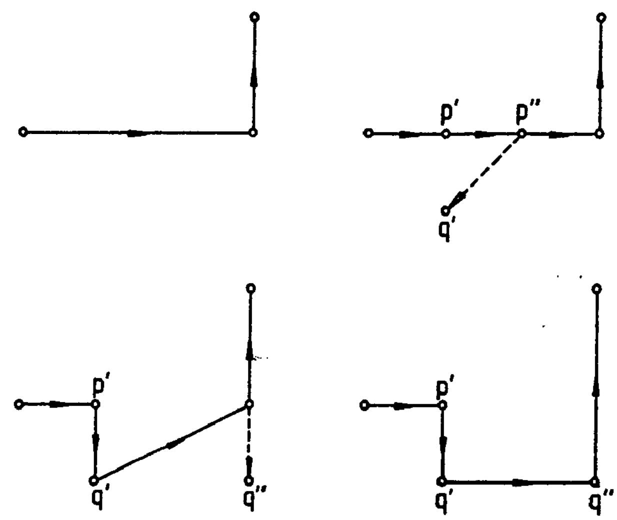

Figure 6

4.2 Isothetic Normal Forms

Let P be a RIP with chain-sequence $C = (v_0, \ldots, v_{n-1}), n \geq 2$. The *signature* of C is the word $\sigma(C) \in \{-1, 1\}^*$ of length $n-1$ defined by $\sigma(C) = \sigma(v_0, v_1) \ldots \sigma(v_{n-2}, v_{n-1})$, where $\sigma(v_i, v_{i+1})$ is the signum of the angle $(\pm\pi/2)$ between v_i and $v_{i+1}, 0 \leq i < n$. The *cyclic signature* of C is obtained by appending the symbol $\sigma(v_{n-1}, v_0)$ to $\sigma(C)$. The *signature class* of P is the equivalence class — under cyclic shifts — of the cyclic signature of C. If the signature class of P is represented by $\sigma_0 \ldots \sigma_{n-1}$ the winding number of P is $\left(\sum_{0 \leq i < n} \sigma_i\right)/4$.

For an integer $m \neq 0$ let Σ_m be the equivalence class of 1^{4m}, if $m > 0$, and of $(-1)^{-4m}$, if $m < 0$. Σ_0

is the equivalence class of $1^3(-1)^3$. It is not hard to construct a RIP with signature class Σ_m for any integer m. The following result shows that such polygons are determined completely by their signature class.

Lemma 4.2 *Let P and P' be RIP's with signature class Σ_m. The chain-sequences of P and P' can be transformed into each other via a sequence of $O(|m|+1)$ transformations of type $IT0$. Moreover this sequence can be determined in $O(|m|+1)$ time.*

Proof: We consider the case $m > 0$. The other cases are similar. Let $C = (v_0, \ldots, v_{n-1}), n = 4m$, be a chain-sequence of P. Then $\text{sign}(v_i) = -\text{sign}(v_{i+2})$ for $0 \le i < n-2$, since all exterior angles of P are $\pi/2$. Let $C' = (v'_0, \ldots, v'_{n-1})$ be a chain-sequence of P' with v'_0 and v_0 parallel and $\text{sign}(v_0) = \text{sign}(v'_0)$. Then $\text{sign}(v_i) = \text{sign}(v'_i)$ for $0 \le i < n$. Note that C' can be obtained from any chain-sequence representing P' in constant time via at most three simple cyclic shifts. The following algorithm transforms P into P'' with chain-sequence $(v'_0, v_1, v'_2, v_3, \ldots, v'_{n-2}, v_{n-1})$ in $O(m)$ time. Its invariant is:

$$I \ : \ 0 \le i \le n-2 \wedge (\forall j : 0 \le j < i : v_j = v'_j).$$

$i := 0$;
while $i \ne n-2$ **do**
  if $\text{sign}(v_i + v_{i+2} - v'_i) \ne \text{sign}(v_{i+2})$ **then**
    $v_{i+2}, v_{i+4} := v_{i+2} - v_i, v_{i+4} + v_i$
    {transformation of type $IT0$}
  endif; $\{\text{sign}(v_i + v_{i+2} - v'_i) = \text{sign}(v_{i+2})\}$
  $v_i, v_{i+2} := v'_i, v_{i+2} + v_i - v'_i$;
  {transformation of type $IT0$}
  $i := i + 2$
endwhile.

The only subtlety in checking the invariance of I is the fact that the guard of the if-statement is false when $i = n-4$, since $I \wedge i = n-4 \Rightarrow v_{n-4} + v_{n-2} = v'_{n-4} + v'_{n-2}$ (use $\sum_{0 \le j < n/2} v_{2j} = 0$). Similarly $I \wedge i = n-2 \Rightarrow v_{n-2} = v'_{n-2}$. A similar algorithm is used to transform P'' into P'. □

Corollary 4.3 *Let P and P' be RIP's with signature class Σ_m. Then there is a sequence of $O(|m|+1)$ transformations of type $T2$ taking P into P'. This sequence can be determined in $O(|m|+1)$ time.*

Proof: In view of lemmas 4.1 and 4.2 we may assume that the chain-sequences of P and P' are equal, i.e. P and P' coincide up to a translation. Decomposing this translation into two translations parallel to the x-and y-axis it is easy to transform P into P'. □

4.3 Normal Form Reduction of an RIC

Next we show how the cyclic signature of a RIC C is used to transform C into a shorter chain-sequence. To this end we derive three elementary reductions of subsequences of C of length at most six. A subsequence is a sequence of successive elements of C.

Lemma 4.4 *Let C be a RIC containing a subsequence $C_0 = (v_0, v_1, v_2, v_3)$ with $\sigma(C_0) = 1(-1)1$ or $\sigma(C_0) = (-1)1(-1)$. Then C can be reduced to C' obtained from C by replacing C_0 with $C'_0 = (v_0 + v_2, v_1 + v_3)$ via one transformation of type $IT2$. Moreover $\sigma(C'_0) = 1$ or $\sigma(C'_0) = (-1)$, respectively.*

Proof: The expression for $\sigma(C_0)$ implies that $\text{sign}(v_0) = \text{sign}(v_2)$ and $\text{sign}(v_1) = \text{sign}(v_3)$. Therefore C'_0 transforms into C_0 via a transformation of type $IT1$ with parameters $v_0/(v_0+v_2), v_1/(v_1+v_3) \in \Lambda_0$. □

Lemma 4.5 *Let C be a RIC containing a subsequence $C_0 = (v_0, v_1, v_2, v_3, v_4)$ with $\sigma(C_0) = 1(-1)^2 1$ or $\sigma(C_0) = (-1)1^2(-1)$, and $v_1 + v_3 \ne 0$. Then C can be reduced to C' obtained from C by replacing C_0 with $C'_0 = (v_0, v_1 + v_3, v_2 + v_4)$ via at most two transformations of type $IT0$ or $IT2$. Moreover $\sigma(C_0) = 1(-1)$ or $\sigma(C'_0) = (-1)1$.*

Proof: The expression for $\sigma(C_0)$ implies that $\text{sign}(v_0) = \text{sign}(v_2) = \text{sign}(v_4)$ and $\text{sign}(v_1) = -\text{sign}(v_3)$. We distinguish two cases.

(i) $\text{sign}(v_1 + v_3) = \text{sign}(v_1)$. Let $\lambda_0 = v_1/(v_1 + v_3)$ and $\lambda_1 = v_2/(v_2 + v_4)$. Then $(\lambda_0, \lambda_1) \in \Lambda_2$ and C'_0 transforms into C_0 via a transformation of type $IT1$ with parameters (λ_0, λ_1). It is easy to check that $\sigma(C'_0)$ is the prefix of $\sigma(C_0)$ of length 2.

(ii) $\text{sign}(v_1 + v_3) = -\text{sign}(v_1)$. Let $\lambda_0 = v_0/(v_0 + v_2)$ and $\lambda_1 = v_1/(v_1 + v_3)$, and define $C''_0 = (v_0 + v_2, v_1 + v_3, v_4)$. Then $(\lambda_0, \lambda_1) \in \Lambda_1$ and the transformation of type $IT1$ with parameters (λ_0, λ_1) takes C''_0 into C'_0. In this case $\sigma(C'_0)$ is the suffix of $\sigma(C_0)$ of length 2.

□

Remark 4.6 If $v_1 + v_3 = 0$ then C_0 can be reduced to $(v_0 + v_2 + v_4)$.

Lemma 4.7 *Let C be a RIC with subsequence $C_0 = (v_0, \ldots, v_6)$ for which $\sigma(C_0) = 1^3(-1)^3$ or $\sigma(C_0) = (-1)^3 1^3$, while $v_1 + v_3 + v_5 \ne 0$. Then C can be*

$$
\begin{aligned}
&C_1\\
\mapsto\quad &\{IT1, \text{parameters}(v_3/2, v_3/v_1) \in \Lambda_1\}\\
C_2 \;=\quad &(\tfrac{1}{2}(v_0+v_6), v_3, \tfrac{1}{2}(v_0+v_6), v_1-v_3, v_2-\tfrac{1}{2}v_6, v_3, v_4, v_5, \tfrac{1}{2}v_6)\\
\mapsto\quad &\{IT2, \text{ parameters } (v_1/(v_1-v_3), (v_2-\tfrac{1}{2}v_6)/(v_2+v_4-\tfrac{1}{2}v_6)) \in \Lambda_2\}\\
C_3 \;=\quad &(\tfrac{1}{2}(v_0+v_6), v_3, \tfrac{1}{2}(v_0+v_6), v_1, v_2+v_4-\tfrac{1}{2}v_6, v_5, \tfrac{1}{2}v_6)\\
\mapsto\quad &\{IT1, \text{ parameters } ((v_1+v_3+v_5)/(2v_3), 2(v_2+v_4)/(v_0+v_6)) \in \Lambda_1\}\\
C_4 \;=\quad &(\tfrac{1}{2}(v_0+v_6), \tfrac{1}{2}(v_1+v_3+v_5), v_2+v_4, \tfrac{1}{2}(-v_1+v_3-v_5), \tfrac{1}{2}v_0-v_2-v_4+\tfrac{1}{2}v_6, v_1, v_2+v_4-\tfrac{1}{2}v_6, v_5, \tfrac{1}{2}v_6)\\
\mapsto\quad &\{IT2, \text{ parameters } ((v_0-2v_2-2v_4+v_6)/v_0, v_1/(v_1+v_5)) \in \Lambda_2\}\\
C_5 \;=\quad &(\tfrac{1}{2}(v_0+v_6), \tfrac{1}{2}(v_1+v_3+v_5), v_2+v_4, \tfrac{1}{2}(-v_1+v_3-v_5), \tfrac{1}{2}v_0, v_1+v_5, \tfrac{1}{2}v_6)\\
\mapsto\quad &\{IT2, \text{ parameters } ((-v_1+v_3-v_5)/(v_1+v_3+v_5), v_0/(v_0+v_6)) \in \Lambda_2\}\\
&C'_0\;.
\end{aligned}
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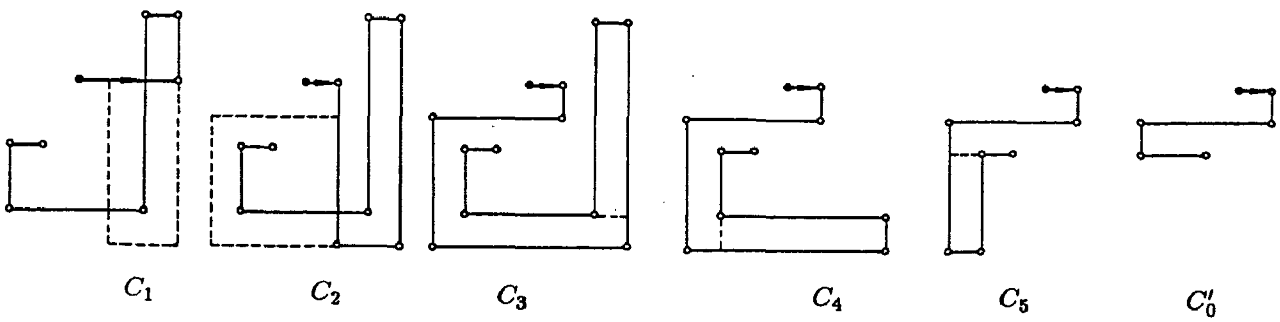

Figure 7

reduced to C' obtained from C by replacing C_0 with $C'_0 = (\frac{1}{2}(v_0+v_6), \frac{1}{2}(v_1+v_3+v_5), v_2+v_4, \frac{1}{2}(v_1+v_3+v_5), \frac{1}{2}(v_0+v_6))$ via at most eight transformations of type $IT0$, $IT1$ and $IT2$. Moreover $\sigma(C'_0) = 1^2(-1)^2$ or $\sigma(C'_0) = (-1)^2 1^2$.

Proof: The expression for $\sigma(C_0)$ implies $\text{sign}(v_1) = -\text{sign}(v_3) = \text{sign}(v_5)$ and $\text{sign}(v_0) = -\text{sign}(v_2) = -\text{sign}(v_4) = \text{sign}(v_6)$. We distinguish two cases:

(i) $\text{sign}(v_1+v_3+v_5) = \text{sign}(v_3)$. Three transformations of type $IT0$ will bring C_0 into the form $C_1 = (v_0+v_6, v_1, v_2-\frac{1}{2}v_6, v_3, v_4, v_5, \frac{1}{2}v_6)$, cf. Figure 7. Next we apply a sequence of five transformations of type $IT1$ and $IT2$. We indicate the type and the parameters and give the transformed chain-sequences.

(ii) $\text{sign}(v_1+v_3+v_5) = -\text{sign}(v_3)$. First we apply two transformations of type $IT0$ to bring C_0 into the form $(v_0, v_1, v_2, -v_1+v_3, v_4, v_1+v_5, v_6)$ and subsequently into the form $C_1 = (v_0, \frac{1}{2}(v_1+v_3+v_5), v_2, \frac{1}{2}(-v_1+v_3-v_5), v_4, v_1+v_5, v_6)$. To C_1 we apply a transformation of type $IT2$ with parameters $(v_2/(v_2+v_4), (-v_1+v_3-v_5)/(v_1+v_3+v_5)) \in \Lambda_1$ to get $C_2 = (v_0, \frac{1}{2}(v_1+v_3+v_5), v_2+v_4, \frac{1}{2}(v_1+v_3+v_5), v_6)$. Three transformations of type $IT0$ finally take C_2 into C'_0. See Figure 8. □

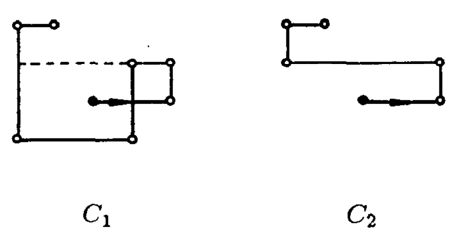

Figure 8

Any subword of a signature σ of the form $1(-1)$ or $(-1)1$ will be called a *dip* of σ. Lemmas 4.4, 4.5 and 4.7 show that the cyclic signature of the chain-sequence C suggests which part of C can be reduced. Any reduction of the type occurring in Lemma 4.4, 4.5 and 4.7 will be called *admissible*. To achieve *linear* reduction time every admissible reduction will take place at the *left-most* dip of σ. However, such a reduction is not always possible without affecting some suffix of σ—and hence of C. Consider e.g. $\sigma = 1(-1)^{10}\ldots$ To eliminate this problem we transform C to guarantee that the left-most dip of σ is preceded by a sufficient number of 1's. This property is an invariant of the algorithm. To describe this modification let $\sigma = \sigma_0 \ldots \sigma_{n-1}$. Then

$$m(\sigma) = \min_{0<k\leq n} \left(\sum_{0\leq i<k} \sigma_i \right)$$

is the minimum of all sums of prefixes of σ.

Lemma 4.8 *Suppose $\sum_{0\leq i<n} \sigma_i = c \geq 0$. Then there is a word σ' obtained from σ via cyclic shift such that $m(\sigma') \geq 0$, if $c = 0$, and $m(\sigma') \geq 1$, if $c \geq 1$. Moreover σ' can be determined in $O(n)$ time.*

Proof: Let $\tau_i = n\sigma_i - c$ for $0 \leq i < n$. Define τ_i for all integers i by $\tau_i = \tau_{i \underline{mod} n}$. Then $\sum_{i\in I} \tau_i = 0$ for every interval $I \subset Z$ with $|I| = n$. Let $t_k = \sum_{0\leq i<k} \tau_i$, then $t_{k+n} = t_k$ for all k. Hence $\min_k t_k$ exists. Let $k0$, with $0 \leq k0 < n$, be such that $t_{k0} = \min_k t_k$, then $\sum_{0\leq i<k} \tau_{k0+i} = t_{k0-k} - t_{k0} \geq 0$, so $n.\sum_{0\leq i<k} \sigma_{k0+i} - kc \geq 0$ for all k. Therefore for all positive integers k we have $\sum_{0\leq i<k} \sigma_{k0+i} \geq 0$, if $c = 0$, and $\sum_{0\leq i<k} \sigma_{k0+i} \geq 1$, if $c \geq 1$. Obviously $k0$ can be determined in $O(n)$ time. □

Remark 4.9 If $c = 1$ the number $k0$ is even unique. Our proof is an algebraic version of the geometric proof of this result in [GKP], where it is referred to as Raney's lemma.

Next consider a reduced isothetic polygon P. We first deal with the case $w(P) > 0$. In view of the preceding lemma we may assume that the chain-sequence C of P has cyclic signature $\sigma_C = \sigma_0 \ldots \sigma_{n-1}$ with $m(\sigma_C) \geq 1$. We may also assume that $\sigma_{n-1} = -1$. First we modify the chain-sequence by applying a transformation of type $IT1$ yielding a chain-sequence with cyclic signature $1\sigma_C(-1) = \sigma(-1)^2$, see Figure 9.
The crucial property of σ is: $m(\sigma) \geq 2$.

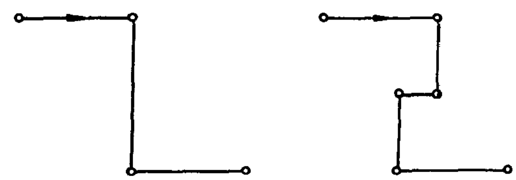

Figure 9

Lemma 4.10 *Suppose σ contains at least two dips or at least three symbols -1. Then at most three admissible reductions of σ at the left-most dip yield a chain-sequence with cyclic signature $\sigma'(-1)^2$, with length $(\sigma') \leq$ length $(\sigma) - 2$ and $m(\sigma') \geq 2$.*

Proof: σ is of the form $1^a(-1)^b\tau$, with $b \geq 3$, or $1^a(-1)^b 1\tau$, with $b = 1$ or $b = 2$. In the latter case lemma 4.4 or 4.5 yields a transformation reducing σ to σ' with the properties as stated. In the former case we apply lemma 4.7 to the part of C corresponding to the subword $1^3(-1)^3$ centered at the left-most dip. Note that $m(\sigma) \geq 2$ and $b \geq 3$ implies $a \geq 5$. The various possible reductions are indicated in Figure 10. The reductions are labelled in accordance with the lemma that justifies them. Note that we may need as many as three reductions, since e.g. $m(1^{a-3}(-1)^2 1^2(-1)^{b-3}\tau) = 0$ if $a = 5$. □

To finish the proof of the main theorem we need a simple lemma.

Lemma 4.11 *There is no reduced isothetic polygon with signature class represented by $(1^2(-1)^2)^l$, $l \geq 1$.*

Proof: Suppose $C = (v_0, \ldots, v_{4l-1})$ has cyclic signature $(1^2(-1)^2)^l$. Then $\text{sign}(v_i) = \text{sign}(v_{i+2})$ either for all odd i or for all even i. Hence $\sum_{0\leq i<4l-1} v_i \neq 0$, a contradiction. □

Proof of theorem 2.2: First consider the case $w(P) > 0$. After $O(n)$ reductions P is reduced to a RIP with signature class $1^{4w(P)+l}(-1)^l$, with $l \leq 4$, cf. lemma 4.10. A constant number of admissible reductions then yields $1^{4w(P)}$. Since all reductions take place at the left-most dip the total time for finding the next reduction is $O(n)$. The case $w(P) < 0$ is completely similar. So assume $w(P) = 0$. Let C be a chain-sequence of P with cyclic signature σ_C satisfying $m(\sigma_C) \geq 0$. Again we transform C in constant time to obtain a cyclic signature $1^2\sigma_C(-1)^2 = \sigma(-1)^3$, with $m(\sigma) \geq 2$. Then the chain-sequence is

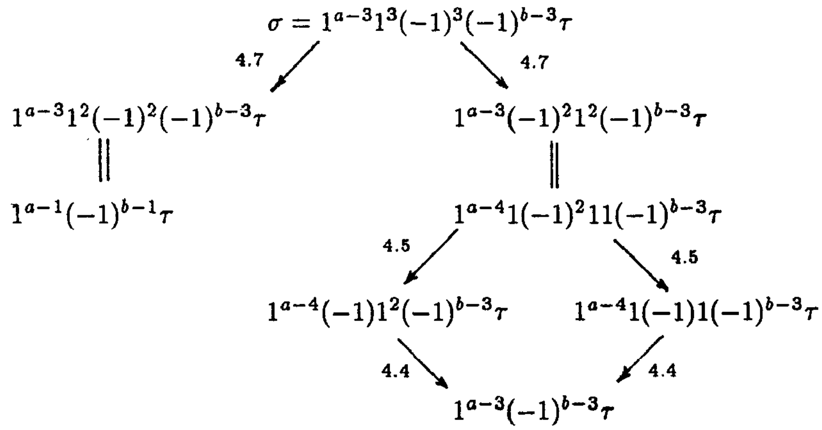

Figure 10

transformed in linear time to obtain a RIC with cyclic signature $1^l(-1)^l$, with $3 \leq l \leq 5$, cf. lemma 4.10. If $l = 5$ the only possible reduction of the cyclic signature is $1^2 1^3(-1)^3(-1)^2 \rightarrow 1^2 1^2(-1)^2(-1)^2$, cf. lemma 4.11. Similarly if $l = 4$ the only possible reduction is $11^3(-1)^3(-1) \rightarrow 11^2(-1)^2(-1)$. So in constant additional time P is reduced to normal form. □

5 Conclusion

We devised a linear algorithm to establish a kink-free deformation of a polygon P to any other polygon with the same winding number. The modifications of P are essentially local. This feature should make the methods amenable for generalization to the context of polygonal curves on piecewise flat surfaces. In the smooth setting Smale derived in [S] a complete invariant for each regular equivalence class by associating with each regular (immersed) curve on a closed surface M an element of the fundamental group of the *unit* tangent bundle of M. We intend to extend the present work to cover this situation as well. As a first step we strive for an algorithm that establishes a homotopy — not necessarily via regular polygons — between polygonal curves that represent the same member of the fundamental group $\pi_1(M)$ of M. This probably amounts to transforming Dehn's method for the word problem in $\pi_1(M)$ into an efficient algorithm (cf. [D]).

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